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FILTRATION IN INHOMOGENEOUS CURVED STRATA WITH A CERTAIN CLASS PERMEABILITY*

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Two-dimensional steady filtration flows of a homogeneous incompressible liquid (obeying D'Arcy's law) in inhomogeneous curved strata with variable permeability is studied. A novel extensive class of permeabilities for which the pressure head function is written in explicit form is obtained and studied. The pressure head function is given in explicit form for flows from sources situated at any point of the stratum.

The basic equations describing two-dimensional flows of steady filtration of a homogeneous incompressible liquid in inhomogeneous curved strata can be written in the form /1/

$$P\frac{\partial\Phi}{\partial x}=\frac{\partial\Psi}{\partial y},\quad P\frac{\partial\Phi}{\partial y}=-\frac{\partial\Psi}{\partial x} \tag{1}$$

Here Φ is the pressure head function, Ψ is the stream function, $P=kM$ is the permeability of the stratum /2/, k is the coefficient of filtration and M is the stratum thickness. We will assume that an isothermal grid is chosen at the surface at the foot of the stratum with coordinates x and y /1/, such that $P=P(y)$. We also assume that $P(y)>0$ and, that a function $\lambda(y)\neq 0$ exists such that the following conditions hold /3/:

$$\lambda P\in\theta_M,\quad \lambda P'\in\theta_M \tag{2}$$

Let the solution of (1) have a singularity at the point (x_0, y_0). Eliminating Ψ we obtain

$$P(y)\left[\frac{\partial^2\Phi}{\partial x^2}+\frac{\partial^2\Phi}{\partial y^2}\right]+P'(y)\frac{\partial\Phi}{\partial y}=0,\quad (x,y)\neq(x_0,y_0) \tag{3}$$

Let us investigate the basic solutions corresponding to the source (sink), since other singularities can be obtained from the source using well-known techniques /1/. Classical methods of obtaining the fundamental solutions for more general equations are well known /4/. However, the solutions obtained by theses methods are very abstract and cannot be used in filtration problems. It was for this reason that $P(y)$ were sought for which the fundamental solution could be obtained in a form suitable for solving specific problems /5, 6/. In /7/ an arbitrarily wide class of $P(y)$ was obtained including the already known cases /5, 6/ as well as new cases in which the solution of the source-sink type is written in explicit form. The present paper deals with the fundamental source-type solutions for such $P(y)$, which were not encountered in the literature until /7/, and completes the investigation carried out in Sect.5 of /7/. To formulate the basic result of this work, we introduce the following definition.

Definition. We shall say that $\Phi\in S'(R^2)$ (/3/, is a fundamental source-type solution of equation (3) at the point (x_0, y_0), if Φ satisfies the equation

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$$\lambda P\left[\frac{\partial^2\Phi}{\partial x^2}+\frac{\partial^2\Phi}{\partial y^2}\right]+\lambda P'\frac{\partial\Phi}{\partial y}=Q\lambda P\delta(x-x_0,y-y_0) \tag{4}$$

Here Q is independent of x and y, and δ is the Dirac function. We will give the basic result of this paper in the form of the following theorem.

Theorem. Let $\mu>0, C>0, \gamma>0$ and

$$P(y)=Z_\mu^2(ih)=[D_1I_\mu(h)+D_2K_\mu(h)]^2 \tag{5}$$

where Z_μ is a cylindrical function /8/ such that $P(y)>0$, I_μ and K_μ are modified Bessel functions /9/, D_1 and D_2 are non-negative constants and

$$h=h(y)=C\exp(\gamma y) \tag{6}$$

Then the solution of (4) is given by the formula ($\theta(z)$ is the Heaviside function)

$$\Phi=\frac{Q}{\pi}\int_0^\infty v\cos[(x-x_0)\xi]\,d\xi \tag{7}$$

$$v=-\frac{1}{\gamma}\sqrt{\frac{P_0}{P}}\{\theta(y_0-y)K_\nu(h_0)I_\nu(h)+\theta(y-y_0)I_\nu(h_0)K_\nu(h)\} \tag{8}$$

$$\nu=(\sqrt{\mu^2\gamma^2+|\xi|^2})/\gamma,\quad P_0=P(y_0),\quad h_0=h(y_0)$$

In proving the theorem we use Olver's theorem /10/ to obtain the asymptotic estimates uniform in ν and h for $I_\nu(h)$ and $K_\nu(h)$, analogous to the asymptotic expansions of these functions for high order /9/. Using these estimates together with (5) and (6), we obtain from (8) an estimate for $|v|$, from which it follows that $v\in S'(R^2)$ and that v is integrable in ξ from $-\infty$ to $+\infty$ for any $y\neq y_0$. This in turn implies the convergence of the integral in (7) and $y\neq y_0$, and the relation $Qv\exp\{ix_0\xi\}=F_x[\Phi]$ where F_x is a Fourier transformation in x /3/. From the last equation it follows that $\Phi\in S'(R^2)$, consequently by virtue of (2) F_x can be applied to both parts of (4). Using (13) and (15) of pt.9, Sect.7, (16) of pt.3, Sect.9 and (19) of pt.4 of /3/, we conclude that (4) is equivalent to the equation

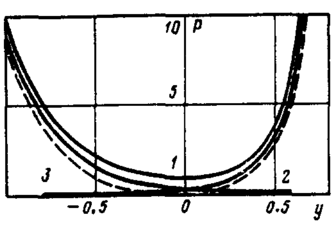
$$\lambda P[v_{yy}''-|\xi|^2v]+\lambda P'v_y'=\lambda P\delta(y-y_0)$$

From /7/ it follows that v defined by (8) satisfies the latter equation. According to (5) and (6), the figure shows the traces of the graphs $P(y)$ for $D_1>0$ and $D_2>0$ (curve *1*), $D_1=0, D_2>0$ (curve *2*), $D_1>0$ and $D_2=0$ (curve *3*) and $D_1=-D_2$ (dashed curve). We note the different rates at which $P(y)$ tends to infinity when $y\to+\infty$ and $y\to-\infty$. Indeed, when $y\to+\infty$ we have from (6) $h\to+\infty$ and /9/

$$P(y)\sim D_1I_\mu(h)\sim\frac{D_1\exp(h)}{\sqrt{2\pi h}}=\frac{D_1\exp(C\exp(\gamma y))}{\sqrt{2\pi C}\exp(\gamma y/2)}$$

At the same time, when $y\to-\infty$, we have from (6) $h\to+0$ and /9/

$$P(y)\sim D_2K_\mu(h)\sim\frac{D_2}{2}\Gamma(\mu)\left(\frac{h}{2}\right)^{-\mu}=\frac{D_2\Gamma(\mu)}{2^{1-\mu}}(C\exp(\gamma y))^{-\mu}$$



This explains the different curvatures of the right and left branch of curve *1*. Curves *2* and *3* are fundamentally different, since they tend to infinity and to zero at different rates. In all these cases the fundamental source-type solution obtained is identical with the stream function, since $P(y)>0$. The case represented by the dashed line is not discussed here.

From the mathematical point of view the theorem given above yields a solution of the Cauchy problem for a new class of equations, while from the mechanical point of view its interest lies in the fact that it can be applied to numerous problems of the theory of filtration describing the conditions of the theorem, and in the simplicity of the results.

We note that the class of functions (5) is sufficiently wide because of the abundance of parameters. Supplementing this with a conformal transformation of the variables x and y (it does not alter the form of the system (1) /6/), we obtain a class of functions which can approximate, with some degree of accuracy, many strata permeabilities encountered in practice. The simplicity of (7) and (8) enables the above theory to be used to study the influence of non-uniformity in the strata permeability in the problem of determining the pressure function in the case of forced filtration towards a well, calculation of the well debit, and in the problems of the displacement of the boundary separating "different colour fluids" /1/. The theorem yields not only the flows corresponding to a point source, but also to a point vortex /12/, and a dipole and multipoles in the intermediate layer. Equations (7) and (8) can be used in analytical studies, or to obtain asymptotic expansions of various kinds /10/.

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